THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 5

Tongou Yang

1. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g \circ f : [a, b] \to \mathbb{R}$ is Riemann integrable. Note this question is easy if we assume that g is Lipschitz.

2. (Optional, MATH 4050 required) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $g : [c, d] \to [a, b]$ be continuous. Is $f \circ g : [c, d] \to \mathbb{R}$ Riemann integrable?

3. Prove that for each $f, g \in \mathcal{R}[a, b]$, we have $fg \in \mathcal{R}[a, b]$.

   Corollary of (1): Let $f \in \mathcal{R}[a, b]$.

   (a) For $p = 1, 2, 3, \ldots$, the function $f^p \in \mathcal{R}[a, b]$.
   (b) $|f| \in \mathcal{R}[a, b]$. Moreover, for $p > 0$, the function $|f|^p \in \mathcal{R}[a, b]$.
   (c) If $f \neq 0$ and $\frac{1}{f}$ is bounded, then $\frac{1}{f} \in \mathcal{R}[a, b]$
   (d) $e^{f(x)}$, $\sin f(x)$,... are in $\mathcal{R}[a, b]$.

   (We know that $\mathcal{R}[a, b]$, the collection of all real valued Riemann integrable functions on $[a, b]$ is a vector space over $\mathbb{R}$. This shows that $(\mathcal{R}[a, b], +, \cdot)$ is a commutative ring with unity and thus an commutative $\mathbb{R}$-algebra.)

4. (**Warning: This topic is for reference only**) For a bounded function $f : [a, b] \to \mathbb{R}$, we define its variation to be

$$\mathrm{Var}(f) := \sup\{\sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)|\},$$

   where the supremum is taken with respect to all partitions $a = x_0 < x_1 < \cdots < x_n = b$, whenever the supremum exists in $\mathbb{R}$. In this case, we say f is of bounded variation on $[a, b]$.

   (a) Let f be of bounded variation on $[a, b]$. Show that f is Riemann integrable on $[a, b]$.
   (b) Let f be monotone. Show that f is of bounded variation, and thus f is Riemann integrable on $[a, b]$. Do the same for Lipschitz continuous functions on $[a, b]$.
   (c) Show that $f := \chi_{\mathbb{Q}\cap[0,1]}$ is not of bounded variation on $[0, 1]$.
   (d) We have shown that the function $f : [0, 1] \to \mathbb{R}$ is Riemann integrable:

$$f(x) := \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ for some } n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

   Is this function of bounded variation?